Estimating the correlation coefficient of bivariate gamma distributions using the maximum likelihood principle and the inference functions for margins

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I. Problem statement

This technical report studies the existence and uniqueness of the maximum likelihood and inference for margins estimators for the parameters of bivariate gamma distributions. The study is first conducted for gamma distributions whose margins have the save shape parameter referred to as mono sensor bivariate gamma distributions. The extension to multivariate multi sensor bivariate gamma distributions is then discussed.

II. Mono-sensor bivariate gamma distributions

A random vector $\mathbf{X} = (X_1, X_2)^T$ is distributed according to an MoMGD on $\mathbb{R}_+^2$ with shape parameter q and scale parameter P if its moment generating function, or Laplace transform, is defined as follows [1]:

$$\psi_{q,P}(\boldsymbol{z}) = E\left(e^{-\sum_{i=1}^{2} X_i z_i}\right) = [P(\boldsymbol{z})]^{-q}, \tag{1}$$

where $\boldsymbol{z} = (z_1, z_2)$, $q \geq 0$ and $P(\boldsymbol{z})$ is the following affine polynomial[1]:

$$P(\boldsymbol{z}) = 1 + p_1 z_1 + p_2 z_2 + p_{12} z_1 z_2, \tag{2}$$

with the following conditions

$$p_1 > 0, \quad p_2 > 0, \quad p_{12} > 0, \quad p_1 p_2 - p_{12} \geq 0. \tag{3}$$

It is important to note that the conditions (3) ensure that (1) is the Laplace transform of a probability distribution defined on $[0, \infty[^2$.

[1] A polynomial $P(\boldsymbol{z})$ where $\boldsymbol{z} = (z_1, \ldots, z_d)$ is *affine* if the one variable polynomial $z_j \mapsto P(\boldsymbol{z})$ can be written $Az_j + B$ (for any $j = 1, \ldots, d$), where A and B are polynomials with respect to the z_i's with $i \neq j$.

A. Margins

The Laplace transform of X_i is obtained by setting $z_j = 0$ for $j \neq i$ in (1). This shows that X_i is distributed according to a univariate gamma distribution with shape parameter q and scale parameter p_i, denoted as $X_i \sim \mathcal{G}(q, p_i)$. Thus, all margins of $\boldsymbol{X}$ are univariate gamma distributions with the same shape parameter q.

B. Probability density function

The pdf of an MoBGD can be expressed as follows (see [2, p. 436] for a similar result)

$$f_{2D}(\mathbf{x}) = \exp\left(-\frac{p_2 x_1 + p_1 x_2}{p_{12}}\right) \frac{x_1^{q-1} x_2^{q-1}}{p_{12}^q \Gamma(q)} f_q(c x_1 x_2) \mathbb{I}_{\mathbb{R}_+^2}(\mathbf{x}), \quad (4)$$

where $\mathbb{I}_{\mathbb{R}_+^2}(\mathbf{x})$ is the indicator function on $[0, \infty[^2$ ($\mathbb{I}_{\mathbb{R}_+^2}(\mathbf{x}) = 1$ if $x_1 > 0, x_2 > 0$ and $\mathbb{I}_{\mathbb{R}_+^2}(\mathbf{x}) = 0$ otherwise), $c = (p_1 p_2 - p_{12})/p_{12}^2$ and $f_q(z)$ is related to the confluent hypergeometric function [2, p. 462] defined by

$$f_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(q+k)}.$$

C. Moments

The moments of a random vector $\mathbf{X}$ can be obtained by differentiating the Laplace transform (1). Straightforward computations allow us to obtain the following results:

$$E[X_1] = q p_1, E[X_2] = q p_2,$$

$$\text{var}(X_1) = q p_1^2, \text{var}(X_2) = q p_2^2$$

$$\text{cov}(X_1, X_2) = q(p_1 p_2 - p_{12}),$$

$$r(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sqrt{\text{var}(X_1)}\sqrt{\text{var}(X_2)}} = \frac{p_1 p_2 - p_{12}}{p_1 p_2}.$$

It is important to note that for a known value of q, an MoBGD is fully characterized by $\boldsymbol{\theta} = (\mathsf{E}[X_1], \mathsf{E}[X_2], r(X_1, X_2))^T$ which will be denoted $\boldsymbol{\theta} = (m_1, m_2, r)^T$ in the remaining of the paper. Indeed, $\boldsymbol{\theta}$ and (p_1, p_2, p_{12}) are obviously related by a one-to-one transformation. Note also that the conditions (3) ensure that the covariance and correlation coefficient of (X_1, X_2) are both positive.

III. Maximum likelihood method for the parameters of a bivariate gamma distribution

A. Principle

The maximum likelihood (ML) method can be applied to estimate $\boldsymbol{\theta}$ since a closed-form expression of the density is available. After removing the terms which do not depend on $\boldsymbol{\theta}$, the log-likelihood function can be written as follows:

$$l(\mathbf{x};\boldsymbol{\theta}) = -nq\log(m_1m_2) - \sum_{j=1}^{2}\frac{nq\overline{x}_j}{m_j(1-r)} - nq\log(1-r) + \sum_{i=1}^{n}\log f_q(cx_1^ix_2^i), \tag{5}$$

where $c = \frac{rq^2}{m_1m_2(1-r)^2}$, and $\overline{x}_j = \frac{1}{n}\sum_{i=1}^{n}x_j^i$ is the sample mean of x_j for $j = 1,2$. By differentiating the log-likelihood with respect to m_1, m_2 and r, and by noting that $f_q'(z) = f_{q+1}(z)$, the following set of equations is obtained

$$\frac{nq\overline{x}_1}{1-r} - nqm_1 - \frac{r}{(1-r)^2}\frac{q^2}{m_2}\Delta = 0, \tag{6}$$

$$\frac{nq\overline{x}_2}{1-r} - nqm_2 - \frac{r}{(1-r)^2}\frac{q^2}{m_1}\Delta = 0, \tag{7}$$

$$\frac{nq\overline{x}_1}{(1-r)m_1} + \frac{nq\overline{x}_2}{(1-r)m_2} - nq - \frac{1+r}{(1-r)^2}\frac{q^2}{m_1m_2}\Delta = 0, \tag{8}$$

where

$$\Delta = \sum_{i=1}^{n}x_1^ix_2^i\frac{f_{q+1}(cx_1^ix_2^i)}{f_q(cx_1^ix_2^i)}. \tag{9}$$

The maximum likelihood estimators (MLEs) of m_1 and m_2 are then easily obtained from these equations:

$$\widehat{m_1}_{\text{ML}} = \overline{x}_1, \quad \widehat{m_2}_{\text{ML}} = \overline{x}_2. \tag{10}$$

After replacing m_1 and m_2 by their MLEs in (8), we can easily show that the MLE of r is obtained by computing the root $r \in [0,1[$ of the following function

$$g(r) = r - 1 + \frac{q}{n\overline{x}_1\overline{x}_2}\left(\sum_{i=1}^{n}x_1^ix_2^i\frac{f_{q+1}(\widehat{c}x_1^ix_2^i)}{f_q(\widehat{c}x_1^ix_2^i)}\right) = 0, \tag{11}$$

where

$$\widehat{c} = \frac{r}{(1-r)^2}\frac{q^2}{\overline{x}_1\overline{x}_2}.$$

B. Concavity of the log-likelihood in the particular case $p_1 = p_2 = p_{12}$

This section focuses on the interesting particular case where the affine polynomial P (2) associated to a bivariate Gamma distribution has equal coefficients:

$$p_1 = p_2 = p_{12} = c. \tag{12}$$

The corresponding bivariate gamma distributions will be referred to as normalized bivariate gamma distribution. Indeed, if (Y_1, Y_2) is distributed according to a bivariate gamma distribution with Laplace transform (2), it can easily be shown that the vector $(X_1, X_2) = (\alpha Y_1, \beta Y_2)$, with $\alpha = \frac{p_2}{p_{12}}$ and $\beta = \frac{p_1}{p_{12}}$, is a distributed according to a normalized bivariate gamma distribution with coefficient $c = \frac{1}{1-r}$, where r is the correlation coefficient of X_1 and X_2. As a consequence, the mean of X_i for $i = \{1, 2\}$ is $m_i = qp_i = q/(1 - r)$ and the log-likelihood of (X_1, X_2) defined in (5) reduces to a function of r only, which can be written (up to an additive constant):

$$l(\mathbf{x}; r) = nq \log(1 - r) + \sum_{i=1}^{n} \log f_q(r x_1^i x_2^i). \tag{13}$$

This section shows that the log-likelihood function $l(\mathbf{x}; r)$ has a unique maximum in $[0, 1[$.

*1) **Concavity of** $l(\mathbf{x}; r)$:* by using the concavity of $x \mapsto \log f_q(x)$ on the interval $[0, +\infty[$ (see proof in the appendix), we can prove that each function $r \mapsto \log f_q(r x_1^i x_2^i)$ for $i = 1 \ldots n$ is a strictly concave function of r in $[0, 1[$. As the function $h : \ x \mapsto nq \log(1 - r)$ is strictly concave in $[0, 1[$ (the reader will check easily that $h''(x) = -nq/(1 - r)^2 < 0$), the log-likelihood $l(\mathbf{x}; r)$ expressed in (13) is also a strictly concave function of r in $[0, 1[$ (as it is the sum of strictly concave functions).

*2) **Unicity of the maximum of** $l(\mathbf{x}; r)$:*

Proposition 3.1: In the normalized case defined by $p_1 = p_2 = p_{12}$, the MLE of r denoted as $\widehat{r}_{ML}$ is unique. Moreover $\widehat{r}_{ML} = 0$ if and only if $\frac{1}{n}\sum_{i=1}^{n} x_1^i x_2^i \leq q^2$. Otherwise $\widehat{r}_{ML}$ is the unique root in $]0, 1[$ of the following score function:

$$g(r) = \frac{\partial l(\mathbf{x}; r)}{\partial r} = \frac{-q}{1 - r} + \frac{1}{n} \sum_{i=1}^{n} x_1^i x_2^i \frac{f_{q+1}(r x_1^i x_2^i)}{f_q(r x_1^i x_2^i)}. \tag{14}$$

Proof: Since $l(\mathbf{x}; r)$ is concave, the score function $g(r)$ is a strictly decreasing function such that

$$g(0) = -q + \frac{1}{nq} \sum_{i=1}^{n} x_1^i x_2^i.$$

Depending on the value of $g(0)$, we have two possible situations:

- $g(0) \leq 0$: in this case the log-likelihood is a strictly decreasing function on $[0, 1[$ and the maximum is reached in $\widehat{r}_{\text{ML}} = 0$. Note that $g(0) \leq 0$ is equivalent to $\frac{1}{n}\sum_{i=1}^{n} x_1^i x_2^i \leq q^2$.
- $g(0) > 0$: it can be shown that $g(r) \to -\infty$ when $r \to 1$ inducing that g is a continuous mapping from $[0, 1[$ to $[g(0), -\infty[$. As a consequence, there exists a unique root of g in $[0, 1[$ which is the MLE of r. This root is the solution of $g(r) = 0$, or equivalently the solution of the following nonlinear relation
$$\frac{\partial l(\mathbf{x}; r)}{\partial r} = \frac{-q}{1-r} + \frac{1}{n}\sum_{i=1}^{n} x_1^i x_2^i \frac{f_{q+1}(r x_1^i x_2^i)}{f_q(r x_1^i x_2^i)} = 0.$$

□

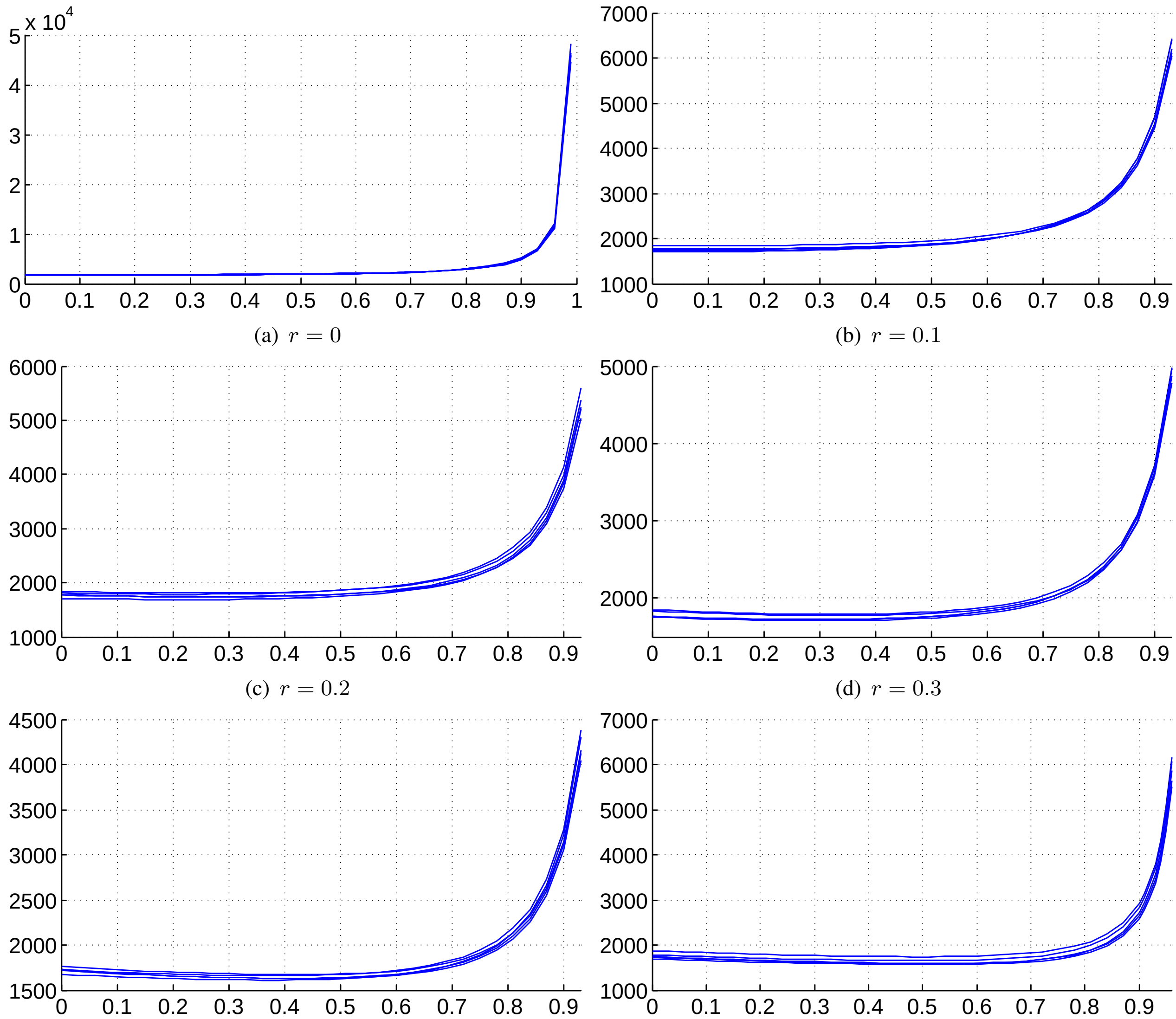
C. *Shape of the optimized criterion*

Unfortunately, the concavity property of the log-likelihood is no longer satisfied in the general case. The shape of the negative log-likelihood for typical values of the data samples (x_1^i, x_2^i), $i = 1, ..., n$, is illustrated in the figures below. For large values of r such that $r = 0.98$, we can observe that the log-likelihood is not concave. However, it can also be seen that any gradient algorithm will converge to the unique minimum of this negative log-likelihood.

IV. MULTI-SENSOR BIVARIATE GAMMA DISTRIBUTIONS

A. *Definition*

A vector $\mathbf{Y} = (Y_1, Y_2)^T$ distributed according to an MuBGD (denoted as $\mathbf{Y} \sim \mathcal{G}(\mathbf{q}, P)$), where $\mathbf{q} = (q_1, q_2)$ and P is an affine polynomial) is constructed from a random vector $\mathbf{X} = (X_1, X_2)^T$ distributed according to an MoBGD whose pdf is denoted as $f_{\mathbf{X}}(\mathbf{x})$ and a random variable $Z \sim \mathcal{G}(q_2 - q_1, p_2)$ independent on $\mathbf{X}$ with pdf $f_Z(z)$. By using the independence assumption between $\mathbf{X}$ and Z, the density of $\mathbf{Y}$ can be expressed as
$$f_{\mathbf{Y}}(\mathbf{y}) = \int f_{\mathbf{X}}(y_1, s) f_Z(y_2 - s) ds. \tag{15}$$
Straightforward computations leads to the following expression:
$$f_{\mathbf{Y}}(\mathbf{y}) = \left(\frac{p_1 p_2}{p_{12}}\right)^{q_1} \frac{y_1^{q_1-1} y_2^{q_2-1}}{p_1^{q_1} p_2^{q_2}} \frac{e^{-\left(\frac{p_2}{p_{12}} y_1 + \frac{p_1}{p_{12}} y_2\right)}}{\Gamma(q_2)\Gamma(q_1)} \Phi_3\left(q_2 - q_1; q_2; c\frac{p_{12}}{p_2} y_2, c y_1 y_2\right), \tag{16}$$
where $c = (p_1 p_2 - p_{12})/p_{12}^2$ and where Φ_3 is the so-called Horn function. The Horn function is one of the twenty convergent confluent hypergeometric series of order two, defined as [3]:

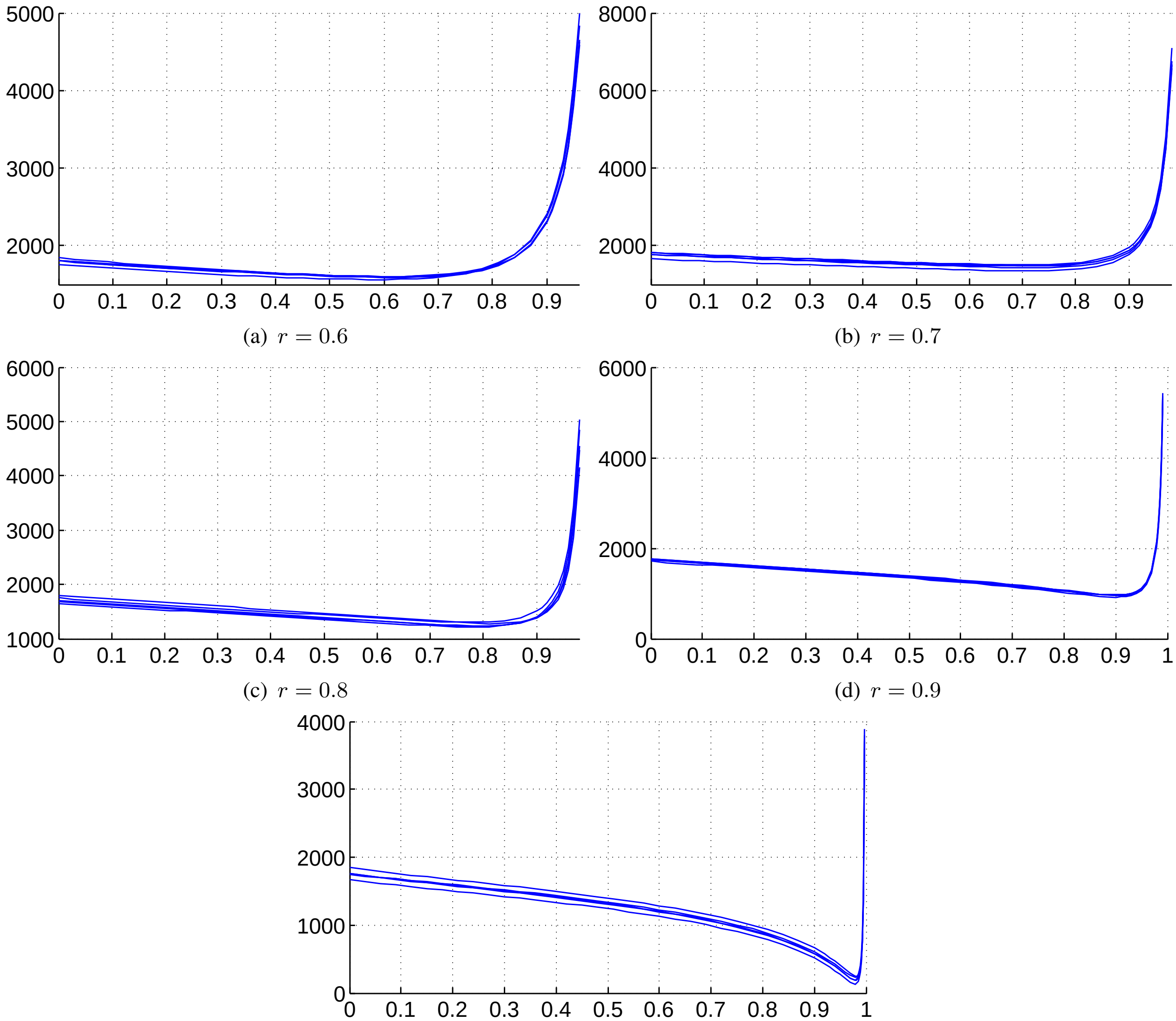


(a) $r = 0$ (b) $r = 0.1$ (c) $r = 0.2$ (d) $r = 0.3$ (e) $r = 0.4$ (f) $r = 0.5$

Fig. 1. Typical plots of the negative log-likelihood versus r for mono-sensor images ($q = 2$, $n = 1000$).

$$\Phi_3(a;b;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m}{(b)_{m+n} m! n!} x^m y^n, \tag{17}$$

where $(a)_m$ is the Pochhammer symbol such that $(a)_0 = 1$ and $(a)_{k+1} = (a+k)(a)_k$ for any positive integer k. It is interesting to note that the relation $f_q(cy_1y_2) = \Phi_3\left(0; q; c\frac{p_{12}}{p_2}y_2, cy_1y_2\right)/\Gamma(q)$ allows one to show that the MuBGD pdf defined in (17) reduces to the MoBGD pdf (4) for $q_1 = q_2 = q$.



(a) $r = 0.6$ (b) $r = 0.7$ (c) $r = 0.8$ (d) $r = 0.9$ (e) $r = 0.98$

Fig. 2. Typical plots of the negative log-likelihood versus r for mono-sensor images ($q = 2$, $n = 1000$).

B. First moments of an MuBGD

The moments of $\mathbf{Y}$ can be obtained from the moments of $\mathbf{X}$ and $\mathbf{Z}$. For instance, by using the independence between $\mathbf{X}$ and $\mathbf{Z}$, the following results can be obtained:

$$E[Y_1] = q_1 p_1, \quad E[Y_2] = q_2 p_2,$$

$$\text{var}(Y_1) = q_1 p_1^2, \quad \text{var}(Y_2) = q_2 p_2^2$$

$$\text{cov}(Y_1, Y_2) = \text{cov}(X_1, X_2) = q_1(p_1 p_2 - p_{12}),$$

$$r(Y_1, Y_2) = \frac{\text{cov}(X_1, Y_2)}{\sqrt{\text{var}(Y_1)}\sqrt{\text{var}(Y_2)}} = \sqrt{\frac{q_1}{q_2}} \frac{p_1 p_2 - p_{12}}{p_1 p_2}.$$

It is interesting to note that the conditions (3) ensure that the correlation coefficient obtained in the bivariate case ($d = 2$) satisfy the constraint $0 \leq r(Y_1, Y_2) \leq \sqrt{q_1/q_2}$. In other words, the normalized correlation coefficient defined by

$$r'(Y_1, Y_2) = \sqrt{\frac{q_2}{q_1}} \quad r(Y_1, Y_2) = \frac{p_1 p_2 - p_{12}}{p_1 p_2},$$

is such that $0 \leq r'(Y_1, Y_2) \leq 1$. Note that for known values of the shape parameters q_1 and q_2, an MuBGD is fully characterized by the parameter vector $\boldsymbol{\theta} = (E[Y_1], E[Y_2], r'(Y_1, Y_2))$, since $\boldsymbol{\theta}$ and (p_1, p_2, p_{12}) are related by a one-to-one transformation.

V. Inference functions for margins for the parameters of a multi-sensor bivariate gamma distribution

The density of an MuBGD (16) can be parametrized by $\boldsymbol{\theta} = (m_1, m_2, r')^T \in \Delta = (0, \infty)^2 \times (0, 1)$. After removing the terms which do not depend on $\boldsymbol{\theta}$, the log-likelihood function of $\boldsymbol{Y}$ can be written

$$l(\boldsymbol{Y}; \boldsymbol{\theta}) = -nq_1 \log(1 - r') - nq_1 \log m_1 - nq_2 \log m_2 - n\frac{q_1}{m_1(1 - r')}\overline{Y}_1 - n\frac{q_2}{m_2(1 - r')}\overline{Y}_2 + \sum_{i=1}^{n} \log \Phi_3 \left(q_2 - q_1; q_2; dY_2^i, cY_1^i Y_2^i\right), \tag{18}$$

where $d = \frac{r' q_2}{m_2(1-r')}$, $\overline{Y}_1 = \frac{1}{n}\sum_{i=1}^{n} Y_1^i$, $\overline{Y}_2 = \frac{1}{n}\sum_{i=1}^{n} Y_2^i$ are the sample means of Y_1 and Y_2 and c defined previously can be expressed as function of $\boldsymbol{\theta}$ using the relation $c = \frac{r' q_1 q_2}{m_1 m_2 (1-r')^2}$.

IFM is a two-stage estimation method whose main ideas can be found for instance in [4, Chapter 10] and are summarized below in the context of MuBGDs:

- estimate the unknown parameters m_1 and m_2 from the marginal distributions of Y_1 and Y_2. This estimation is conducted by maximizing the marginal likelihoods $l(Y_1; m_1)$ and $l(Y_2; m_2)$ wrt m_1 and m_2 respectively,
- estimate the parameter r' by maximizing the joint likelihood $l(\boldsymbol{Y}; \widehat{m_1}_{\text{ML}}, \widehat{m_2}_{\text{ML}}, r')$ wrt r'. Note that the parameters m_1 and m_2 have been replaced in the joint likelihood by their estimates resulting from the first stage of IFM.

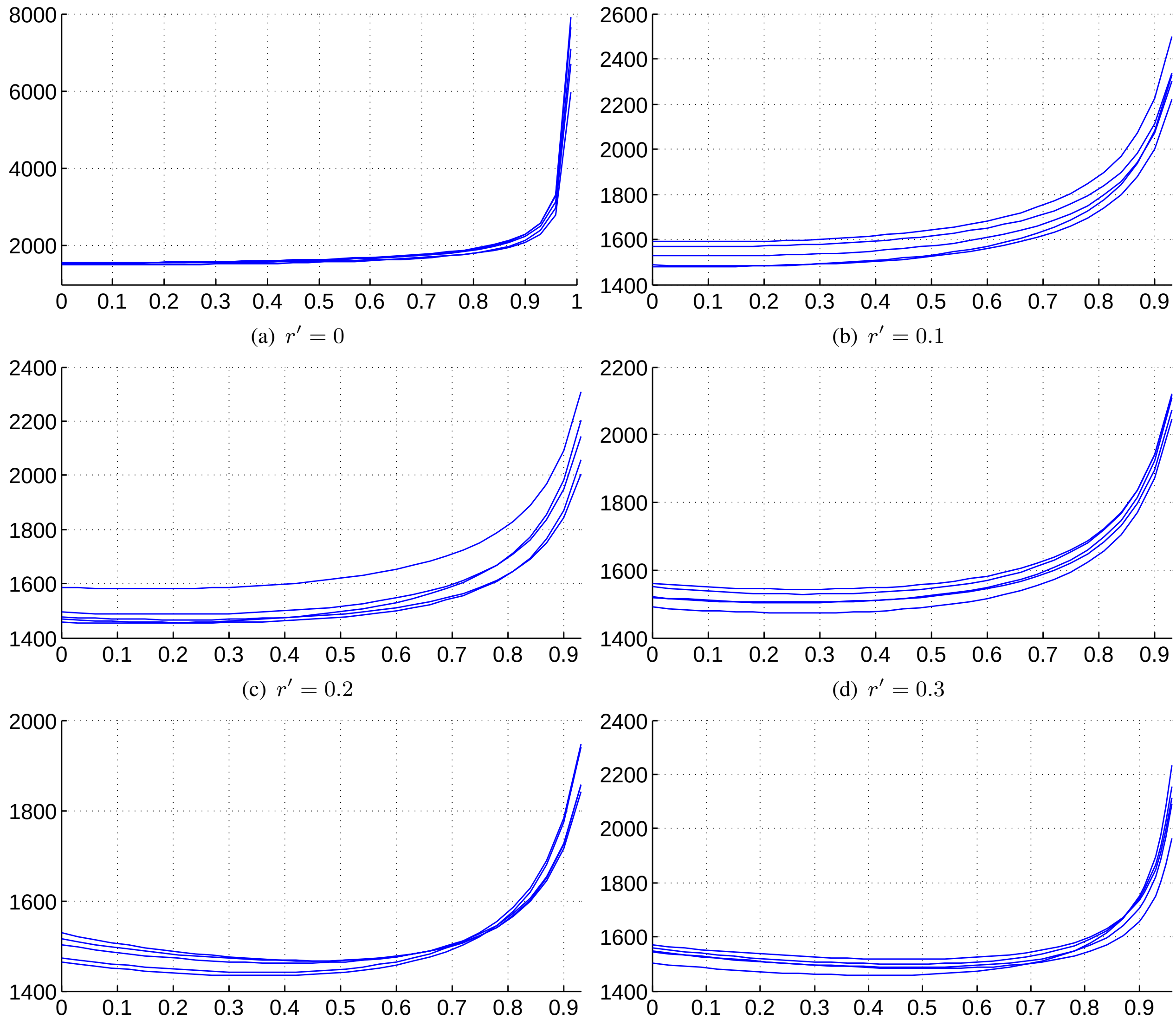
The IFM procedure is often computationally simpler than the ML method which estimates all the parameters simultaneously from the joint likelihood. Indeed, a numerical optimization with several parameters is much more time-consuming compared with several optimizations with fewer parameters. The marginal distributions of an MuBGD are univariate gamma distributions with shape parameters q_i and means m_i, for $i = \{1, 2\}$. Thus, the IFM estimators of m_1, m_2, r' are obtained as a solution of:

$$\boldsymbol{g}(\boldsymbol{Y}; \boldsymbol{\theta}) = \left(\frac{\partial l_1(\mathbf{Y_1}; m_1)}{\partial m_1}, \frac{\partial l_2(\mathbf{Y_2}; m_2)}{\partial m_2}, \frac{\partial l(\boldsymbol{Y}; m_1, m_2, r')}{\partial r'} \right)^T = \mathbf{0}^T \tag{19}$$

where l_i is the marginal log-likelihood function associated to the univariate random variable Y_i, for $i = \{1, 2\}$, and l is the joint log-likelihood defined in (18). The IFM estimators of m_1 and m_2 are classically obtained from the properties of the univariate gamma distribution:

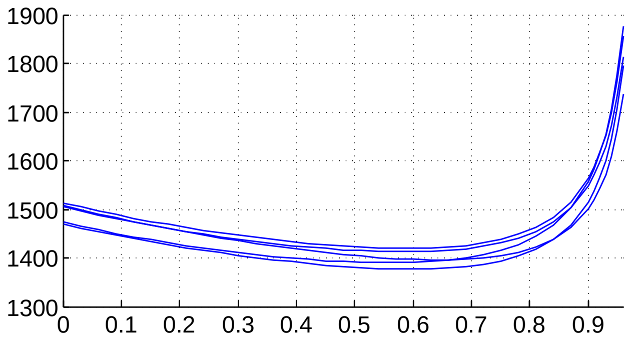
$$\widehat{m_1}_{\text{ IFM}} = \overline{Y}_1, \qquad \widehat{m_2}_{\text{ IFM}} = \overline{Y}_2. \tag{20}$$

The IFM estimator of r' is obtained by replacing m_1 and m_2 by $\overline{Y}_1$ and $\overline{Y}_2$ in (18) and by minimizing the resulting log-likelihood $l(\boldsymbol{Y}; \overline{Y}_1, \overline{Y}_2, r')$ wrt r'. This last minimization is achieved by using a constrained quasi-Newton method (with the constraint $r' \in [0, 1]$), since an analytical expression of the log-likelihood gradient is available. The shape of $l(\mathbf{Y}; \overline{Y}_1, \overline{Y}_2, r')$ for typical values of the data samples (y_1^i, y_2^i), $i = 1, ..., n$ is illustrated in the figures below. It can be clearly seen that any gradient algorithm will converge to the unique minimum of the log-likelihood $l(\mathbf{Y}; \overline{Y}_1, \overline{Y}_2, r')$, as in the mono-sensor case.

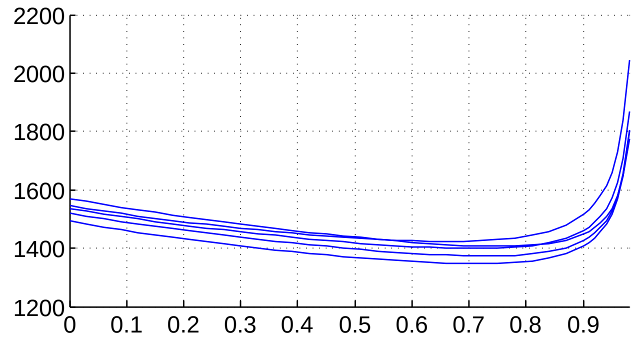


(a) $r' = 0$ (b) $r' = 0.1$ (c) $r' = 0.2$ (d) $r' = 0.3$ (e) $r' = 0.4$ (f) $r' = 0.5$

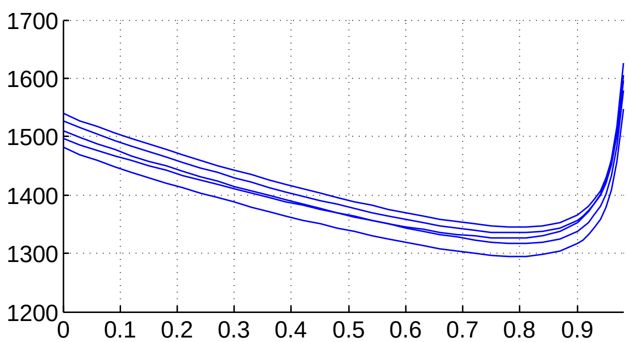
Fig. 3. Typical plots of the negative log-likelihood versus r' ($q_1 = 2$, $q_2 = 4$, $n = 1000$).



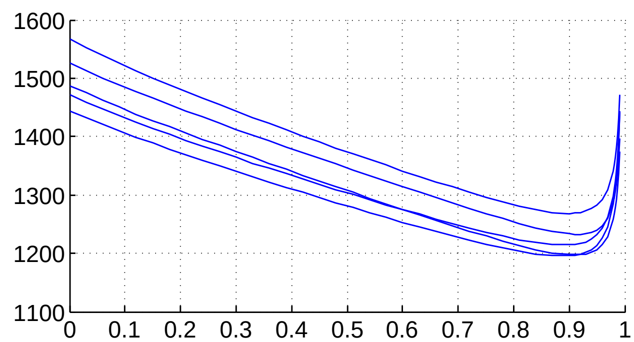
(a) $r' = 0.6$



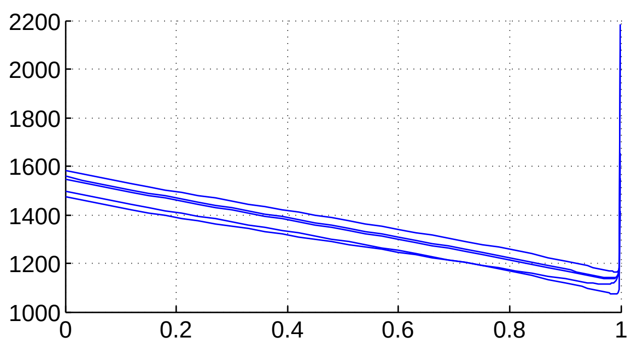
(b) $r' = 0.7$



(c) $r' = 0.8$



(d) $r' = 0.9$



(e) $r' = 0.98$

Fig. 4. Typical plots of the negative log-likelihood versus r' ($q_1 = 2$, $q_2 = 4$, $n = 1000$) for several samples.

VI. APPENDIX: CONCAVITY OF $\log f_q$

The confluent hypergeometric function f_q is such that $f_q'(z) = f_{q+1}(z)$ and $f_q''(z) = f_{q+2}(z)$. Thus the function $\log f_q$ is concave if $f_{q+2}(x)f_q(x) - f_{q+1}^2(x) < 0$ for all $x > 0$, or equivalently

$$f_{q+1}(x)f_{q-1}(x) - f_q^2(x) < 0, \qquad x > 0. \tag{21}$$

Let us denote $u_n(q) = \frac{1}{n!\Gamma(q+n)}$. The coefficient of z^n of the entire function $f_{q+1}(z)f_{q-1}(z) - f_q^2(z)$ is

$$v_n(q) = \sum_{k=0}^{n}[u_k(q+1)u_{n-k}(q-1) - u_k(q)u_{n-k}(q)].$$

A sufficient condition for (21) to be valid is $v_n(q) < 0$ for all $q > 1$. Straightforward computations lead to

$$\begin{aligned}
v_n(q) &= \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\left[\frac{1}{\Gamma(q+k+1)\Gamma(q+n-k-1)} - \frac{1}{\Gamma(q+k)\Gamma(q+n-k)}\right] \\
&= \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\frac{1}{\Gamma(q+k)\Gamma(q+n-k)}\left[\frac{q+n-k-1}{q+k} - 1\right] \\
&= \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\frac{1}{\Gamma(q+k)\Gamma(q+n-k)}\frac{n-2k-1}{q+k} \\
&= \sum_{k=0}^{n} \frac{n-2k-1}{k!(n-k)!}\frac{1}{\Gamma(q+k+1)\Gamma(q+n-k)} \\
&= -\frac{(2n-1)}{\Gamma(q+n+1)\Gamma(q)} + \sum_{k=0}^{n-1} \frac{n-2k-1}{k!(n-k)!}\frac{1}{\Gamma(q+k+1)\Gamma(q+n-k)} \\
&= -\frac{(2n-1)}{\Gamma(q+n+1)\Gamma(q)} + \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{\Gamma(q+k+1)\Gamma(q+n-k)}\left[\frac{n-2k-1}{k!(n-k)!} + \frac{n-2(n-1-k)-1}{(n-1-k)!(k+1)!}\right] \\
&= -\frac{(2n-1)}{\Gamma(q+n+1)\Gamma(q)} - \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{\Gamma(q+k+1)\Gamma(q+n-k)}\left[\frac{(n-2k-1)^2}{(k+1)!(n-k)!}\right] < 0.
\end{aligned}$$

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