ESTIMATING THE POLARIZATION DEGREE OF POLARIMETRIC IMAGES USING MAXIMUM LIKELIHOOD METHODS

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ABSTRACT

This paper shows that the joint distribution of polarimetric intensity images is a multivariate gamma distribution in the case of coherent illumination with fully developed speckle. The parameters of this gamma distribution can be estimated according to the maximum likelihood (ML) principle. Different estimators depending on the number of available polarimetric images are studied. These estimators provide different ways of estimating the degree of polarization (DoP) associated to each pixel of the image. A performance comparison with estimators based on methods of moments shows the interest of the ML method for estimating the DoP of polarimetric images.

1. INTRODUCTION

Polarimetric images provide information regarding the polarimetric nature of the light, which are not available with classical intensity images. The acquisition of polarimetric images is useful in many applications including medical imagery [1] or industrial vision [2]. The increasing interest in this type of imagery is due to the fact that imaged objects modify the polarimetric properties of the light. This modification is linked to the constitution of the objects present in the scene. For example, metallic objects less depolarize the incident light than plastics objects. A way to simply characterize the capacity of the material to polarize or depolarize the light is to compute the scalar DoP [3]. This study focusses on the estimation of the DoP when coherent illumination is used and the speckle is fully developed.

Methods of moments have been recently proposed to estimate the DoP of polarimetric images using several intensity images. In particular, it was shown that the DoP can be estimated using two intensity images only [4]. This paper studies new estimation strategies based on the ML principle. The first part of the paper shows that the joint distribution of polarimetric intensity images is a multivariate gamma distribution. Different ML estimators of the parameters associated to this gamma distribution are then derived. These estimators depend on the number of observed polarimetric images. They provide different ways of estimating the DoP of polarimetric images. The maximum likelihood estimator (MLE) is known to have nice asymptotic properties (such as asymptotic unbiasedness, asymptotic efficiency and asymptotic normality) under appropriate regularity conditions. These properties make the MLE a benchmark to which other suboptimal estimators can be compared. The last part of the paper compares the performance of different DoP estimators based on the ML method and methods of moments.

This paper is organized as follows. Section 2 derives the joint distribution of polarimetric images. The DoP of polarimetric images is defined in Section 3. Section 4 studies ML estimators for the parameters of polarimetric images and the associated DoP. Simulation results are presented in Section 5.

2. STATISTICAL PROPERTIES OF POLARIMETRIC IMAGES

The light can be described by a monochromatic electrical field propagating in the $\mathbf{e}_Z$ direction in an homogeneous and isotropic medium at a given point $\mathbf{r}$ at time t

$$\boldsymbol{E}(\mathbf{r},t) = [A_X(\mathbf{r},t)\mathbf{e}_X + A_Y(\mathbf{r},t)\mathbf{e}_Y]e^{-i2\pi\nu t},$$

where ν is the vibration central frequency and $A_X(\mathbf{r},t)$, $A_Y(\mathbf{r},t)$ are the complex components of the so-called Jones vector denoted as $\boldsymbol{A} = \boldsymbol{A}(\mathbf{r},t)$. The state of polarization of the light can be described by the random behavior of the Jones vector through its covariance matrix

$$\Gamma = \begin{pmatrix} E\left[A_X A_X^*\right] & E\left[A_X A_Y^*\right] \\ E\left[A_Y A_X^*\right] & E\left[A_Y A_Y^*\right] \end{pmatrix} = \begin{pmatrix} a_1 & a_2 + ia_3 \\ a_2 - ia_3 & a_4 \end{pmatrix}, \quad (1)$$

where $E[\cdot]$ is the mathematical expectation and * is the complex conjugate. The covariance matrix Γ is a non negative hermitic matrix whose diagonal terms are the intensity components in the X and Y directions. The cross terms of Γ are complex correlations between the Jones components. If we assume a fully developed speckle, the Jones vector $\boldsymbol{A}$ is a complex circular Gaussian vector whose probability density function (pdf) can be written [3]

$$p(\boldsymbol{A}) = \frac{1}{\pi^2|\Gamma|} \exp\left(-\boldsymbol{A}^\dagger\Gamma^{-1}\boldsymbol{A}\right), \tag{2}$$

where $|\Gamma|$ is the determinant of the matrix Γ and $\dagger$ denotes the conjugate transpose operator. The different components of the covariance matrix Γ can be classically estimated by using four intensity images. The images I_1 and I_2 are obtained by analyzing the light backscattered by the scene in two orthogonal states of polarization. This is done by introducing a polarizer between the scene and the camera, which is parallel or orthogonal to the incident light. The intensity I_3 is obtained by recording the light backscattered in the direction oriented to 45^o from the incident light, by modifying the orientation of the polarizer. Finally the image I_4 is obtained by adding a quarter wave plate allowing to introduce a phase difference of $\lambda/4$ in the previous configuration. As a consequence, the four intensities are related to the components of the Jones vector as follows:

$$\begin{aligned} I_1 &= |A_X|^2, \qquad I_2 = |A_Y|^2, \\ I_3 &= \frac{1}{2}|A_X|^2 + \frac{1}{2}|A_Y|^2 + \mathrm{Re}\left(A_X A_Y^*\right), \\ I_4 &= \frac{1}{2}|A_X|^2 + \frac{1}{2}|A_Y|^2 + \mathrm{Im}\left(A_X A_Y^*\right). \end{aligned} \tag{3}$$

This section derives the joint distribution of the intensity vector $\boldsymbol{I} = (I_1, I_2, I_3, I_4)^T$. For this, it is interesting to note that the matrix $S = \boldsymbol{A}\boldsymbol{A}^\dagger$ is distributed according to a complex Wishart distribution whose Laplace transform is:

$$L_{\boldsymbol{S}}(\boldsymbol{\theta}) = E\{\exp\left[-\mathrm{trace}(\boldsymbol{S}\boldsymbol{\theta})\right]\} = |\mathbb{I}_2 + \Gamma\boldsymbol{\theta}|^{-1}, \tag{4}$$

where $\mathbb{I}_2$ is the 2×2 identity matrix and $\boldsymbol{\theta}$ is a 2×2 matrix ensuring existence of $L(\boldsymbol{\theta})$. By using the following notations

$$\boldsymbol{S} = \begin{pmatrix} s_1 & s_2 + is_3 \\ s_2 - is_3 & s_4 \end{pmatrix} = \begin{pmatrix} |A_X|^2 & A_X A_Y^* \\ A_Y A_X^* & |A_Y|^2 \end{pmatrix},$$

and

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 & \theta_2 + i\theta_3 \\ \theta_2 - i\theta_3 & \theta_4 \end{pmatrix},$$

where $(s_i, \theta_i) \in \mathbb{R}^2$, for $i = 1, ..., 4$, we easily obtain

$$\mathrm{trace}(\boldsymbol{S}\boldsymbol{\theta}) = s_1\theta_1 + 2s_2\theta_2 + 2s_3\theta_3 + s_4\theta_4.$$

The intensity vector $\boldsymbol{I}$ is clearly linearly related to $\boldsymbol{s} = (s_1, ..., s_4)^T$ since

$$\boldsymbol{s} = M\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 \\ -1/2 & -1/2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \boldsymbol{I}. \tag{5}$$

Equations (4) and (5) allow us to obtain the Laplace transform of the intensity vector:

$$L_{\boldsymbol{I}}(\boldsymbol{\theta}) = E\left[\exp\left(-\sum_{j=1}^{4}\theta_j I_j\right)\right] = \frac{1}{P(\boldsymbol{\theta})}, \tag{6}$$

where $P(\boldsymbol{\theta})$ is the following affine polynomial[1]

$$P(\boldsymbol{\theta}) = 1 + \boldsymbol{\alpha}^T\boldsymbol{\theta} + k\left[2\theta_1\theta_2 + \theta_3\theta_4 + (\theta_1 + \theta_2)(\theta_3 + \theta_4)\right],$$

with

$$\begin{aligned} \boldsymbol{\alpha} &= \left[a_1, a_4, \frac{1}{2}(a_1 + 2a_2 + a_4), \frac{1}{2}(a_1 + 2a_3 + a_4)\right]^T, \\ k &= \frac{1}{2}(a_1 a_4 - a_2^2 - a_3^2). \end{aligned}$$

As a consequence, the intensity vector $\boldsymbol{I} = (I_1, I_2, I_3, I_4)^T$ is distributed according to a multivariate gamma distribution (MGD) with shape parameter $q = 1$ and scale parameter P as defined in [5, 6]. Moreover, according to (6), the distribution of the intensity vector $\boldsymbol{I}$ is fully characterized by the parameter vector $\boldsymbol{a} = (a_1, ..., a_4)^T$.

3. DEGREE OF POLARIZATION

The DoP of a given pixel in a polarimetric image is defined by [3, p. 134-136]

$$P^2 = 1 - \frac{4\left[a_1 a_4 - (a_2^2 + a_3^2)\right]}{(a_1 + a_4)^2}. \tag{7}$$

It characterizes the state of polarization of the light. The light is totally depolarized for $P = 0$, totally polarized for $P = 1$ and partially polarized when $P \in]0, 1[$. The estimation of P^2 is important in many practical applications. Since only one realization of the random vector $\boldsymbol{I}$ is available for a given pixel, the images are supposed to be locally stationary and ergodic. These assumptions allow us to build estimates using several neighbor pixels belonging to a so-called estimation window.

The next section studies estimators of the DoP based on several vectors $\boldsymbol{I}^1, \ldots, \boldsymbol{I}^n$ (belonging to the estimation window) distributed according an MGD with Laplace transform $L_{\boldsymbol{I}}(\boldsymbol{\theta})$ defined in (6). These estimators are constructed from estimates of the covariance matrix elements a_i, $i = 1, ..., 4$. Different estimators are studied depending on the number of available polarimetric images, i.e. 2 or 4 polarimetric images. Developing estimation methods based on few images is important to reduce the measurement acquisition time as well as the cost of the imagery system, which has motivated this study.

[1] A polynomial $P(\mathbf{z})$ where $\mathbf{z} = (z_1, \ldots, z_d)$ is *affine* if the one variable polynomial $z_j \mapsto P(\mathbf{z})$ can be written $Az_j + B$ (for any $j = 1, \ldots, d$), where A and B are polynomials with respect to the z_i's for $i \neq j$.

4. ML METHOD

4.1. Using 4 Images

Straightforward computations using (3) show that the intensity vector $\boldsymbol{I}$ belongs to a cone whose equation is:

$$\left[I_3 - \frac{I_1+I_2}{2}\right]^2 + \left[I_4 - \frac{I_1+I_2}{2}\right]^2 = I_1 I_2.$$

Consequently, the distribution of $\boldsymbol{I}$ is singular and defined on this cone. The distribution of $\boldsymbol{I}$ belongs to a natural exponential family (see appendix A for details). As a consequence, the MLE of $\boldsymbol{\alpha} = E[\boldsymbol{I}]$ is [7]

$$\widehat{\boldsymbol{\alpha}}_{\text{ML}} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{I}^j.$$

The mean of the intensity vector is related to the vector $\boldsymbol{a}$ as follows

$$\begin{aligned}\alpha_1 &= E[I_1] = a_1, \alpha_3 = E[I_3] = \frac{1}{2}\left(a_1 + a_4 + 2a_2\right),\\ \alpha_2 &= E[I_2] = a_4, \alpha_4 = E[I_4] = \frac{1}{2}\left(a_1 + a_4 + 2a_3\right).\end{aligned}$$

The functional invariance principle can then be used to derive the MLE of $\boldsymbol{a}$:

$$\widehat{\boldsymbol{a}}_{\text{ML}} = M\widehat{\boldsymbol{\alpha}}_{\text{ML}}.$$

Using $\widehat{\boldsymbol{\alpha}}_{\text{ML}} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{I}^j$, it can be proved that the MLE of $\boldsymbol{a}$ is unbiased and efficient providing an optimal estimation of $\boldsymbol{a}$. These MLEs are then plugged into (7) yielding the following DoP estimator based on 4 polarimetric images:

$$\widehat{P}_4^2 = 1 - \frac{4\left[\widehat{a}_1\widehat{a}_4 - (\widehat{a}_2^2 + \widehat{a}_3^2)\right]}{(\widehat{a}_1 + \widehat{a}_4)^2},$$

where $\widehat{a}_i$ is the ith component of the vector $\widehat{\boldsymbol{a}}_{\text{ML}}$ for $i = 1, \ldots, 4$. The asymptotic variance of this estimator $\widehat{P}_4^2$ can be determined and will be given later (see (12)).

4.2. Using 2 images

A moment estimator of P^2 based on two intensity images I_1 and I_2 was recently studied in [4]. This section derives a new DoP estimator based on the MLE of $\underline{\boldsymbol{I}} = (I_1, I_2)^T$. The distribution of $\underline{\boldsymbol{I}}$ can be obtained by its Laplace transform:

$$L_{\underline{\boldsymbol{I}}}(\underline{\boldsymbol{\theta}}) = E\left[\exp\left(-\sum_{j=1}^{2}\theta_j I_j\right)\right] = \frac{1}{\underline{P}(\underline{\boldsymbol{\theta}})}, \qquad (8)$$

where $\underline{P}(\underline{\boldsymbol{\theta}}) = 1 + \alpha_1\theta_1 + \alpha_2\theta_2 + 2k\theta_1\theta_2$ and $\underline{\boldsymbol{\theta}} = (\theta_1, \theta_2)^T$. By recalling that $\alpha_1 = a_1$, $\alpha_2 = a_4$ and $k = \frac{1}{2}(a_1 a_4 - a_2^2 - a_3^2)$, it can be seen that the distribution of $\underline{\boldsymbol{I}}$ is parametrized by three parameters a_1, a_4 and $r = a_2^2 + a_3^2$. As a consequence, one can think of estimating these three parameters by using the ML method. The pdf of the bivariate gamma distribution having the Laplace transform $L_{\underline{\boldsymbol{I}}}(\underline{\boldsymbol{\theta}})$ has been defined in [6]:

$$p(\underline{\boldsymbol{I}}) = \frac{1}{2k}\exp\left(-\frac{a_4}{2k}I_1 - \frac{a_1}{2k}I_2\right) f_1(\nu I_1 I_2)\mathbb{I}_{\mathbb{R}_+^2}(\underline{\boldsymbol{I}}), \qquad (9)$$

with $\nu = \frac{1}{4k^2}(a_1 a_4 - 2k)$. By differentiating the log-likelihood associated to this pdf with respect to a_1, a_4 and r, we obtain:

$$\begin{pmatrix}\widehat{a}_1\\ \widehat{a}_4\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\widehat{\alpha}_1\\ \widehat{\alpha}_2\end{pmatrix},$$

with $\widehat{\alpha}_l = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{I}_l^j$ for $l = 1, 2$. It can be shown that the parameter r can be estimated by computing the root of the following nonlinear equation:

$$\widehat{a}_1\widehat{a}_4 - r - \frac{1}{n}\sum_{j=1}^{n} I_1^j I_2^j \frac{f_2\left(\frac{r I_1^j I_2^j}{(\widehat{a}_1\widehat{a}_4 - r)^2}\right)}{f_1\left(\frac{r I_1^j I_2^j}{(\widehat{a}_1\widehat{a}_4 - r)^2}\right)} = 0. \qquad (10)$$

The root of this non-linear equation can be computed using a Newton-Raphson procedure. Note that the convergence of this numerical procedure has been studied in [8] for specific bivariate distributions. The estimates of a_1, a_4 and r are then plugged into (7) yielding an estimate of the DoP based on 2 polarimetric images:

$$\underline{P}_2^2 = 1 - \frac{4\left[\widehat{a}_1\widehat{a}_4 - \underline{r}\right]}{(\widehat{a}_1 + \widehat{a}_4)^2},$$

The asymptotic covariance matrix of the resulting estimator of the unknown parameter vector $\boldsymbol{\eta} = (a_1, a_4, r)^T$ can be obtained from the asymptotically efficiency property of the MLE under mild regularity conditions. Thus, the asymptotic covariance matrix of the MLE equals the Cramer Rao Lower Bound (CRLB), which is defined as the inverse of the following Fisher information matrix:

$$F_2(\boldsymbol{\eta}) = -E\left[\frac{\partial^2 \log p(\underline{\boldsymbol{I}};\boldsymbol{\eta})}{\partial\boldsymbol{\eta}\partial\boldsymbol{\eta}^T}\right].$$

However, this computation is difficult because of the term $\log f_{\frac{1}{2}}$ appearing in the log-density. In such situation, it is very usual to approximate the expectations by using Monte Carlo methods. More specifically, this approach consists of approximating the elements of the Fisher information matrix (FIM) $F_2(\boldsymbol{\eta})$ as follows

$$\left[F_2(\boldsymbol{\eta})\right]_{ij} \simeq -\frac{1}{N}\sum_{k=1}^{N}\frac{\partial^2 \log p(\mathbf{x}_k)}{\partial\eta_i\partial\eta_j},$$

where $\mathbf{x}_k$ is distributed according to a bivariate gamma distribution whose pdf is defined in (9) and N is the number of

Monte Carlo runs. The asymptotic variance of the estimator $\underline{P}_2^2$ is then obtained as follows:

$$\text{var}\left(\underline{P}_2^2\right) = G_2^T F_2^{-1} G_2,$$

where G_2 is the gradient of the transformation from (a_1, a_4, r) to P^2, i.e. $G_2 = \left(\frac{\partial P^2}{\partial a_1}, \frac{\partial P^2}{\partial a_4}, \frac{\partial P^2}{\partial r}\right)^T$.

5. METHOD OF MOMENTS

In order to appreciate the performance of the MLEs derived above, this section studies estimators based on the classical method of moments.

5.1. Using 4 images

When four polarimetric images are available, the moment estimator of $\boldsymbol{a}$ denoted as $\widehat{\boldsymbol{a}}_{\text{Mo}}$ is also the MLE derived in section 4.1:

$$\widehat{\boldsymbol{a}}_{\text{Mo}} = M\widehat{\boldsymbol{\alpha}}_{\text{Mo}},$$

where $\widehat{\boldsymbol{\alpha}}_{\text{Mom}} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{I}^j$. Therefore the covariance matrix of $\widehat{\boldsymbol{a}}_{\text{Mo}} = \widehat{\boldsymbol{a}}_{\text{ML}}$ is:

$$\text{cov}\left(\widehat{\boldsymbol{a}}_{\text{Mo}}\right) = \frac{1}{n} M \text{cov}\left(\boldsymbol{I}\right) M^T,$$

where $\text{cov}\left(\boldsymbol{I}\right)$ is the covariance matrix of $\boldsymbol{I}$. The matrix $\text{cov}\left(\boldsymbol{I}\right)$ can be classically computed by differentiating the Laplace transform of the vector $\boldsymbol{I}$ (6). Straightforward computations lead to the following result:

$$\text{cov}\left(\widehat{\boldsymbol{a}}_{\text{Mo}}\right) = \frac{1}{n}\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 & a_2^2 + a_3^2 \\ a_1a_2 & c_{2,2} & a_2a_3 & a_4a_2 \\ a_1a_3 & a_2a_3 & c_{3,3} & a_4a_3 \\ a_2^2 + a_3^2 & a_4a_2 & a_4a_3 & a_4^2 \end{pmatrix}, \quad (11)$$

with $c_{2,2} = (a_1a_4 + a_2^2 - a_3^2)/2$ and $c_{3,3} = (a_1a_4 - a_2^2 + a_3^2)/2$. As a consequence, the asymptotic variance of the DoP estimator $\widehat{P}_4^2$ is:

$$\text{var}_\text{A}\left(\widehat{P}_4^2\right) = G_4^T \text{cov}\left(\widehat{\boldsymbol{a}}_{\text{Mo}}\right) G_4 = \frac{2(1-P^2)^2 P^2}{n}, \quad (12)$$

where G_4 is the gradient of the transformation from $\boldsymbol{a}$ to P^2. Note that the asymptotic variance of $\widehat{P}_4^2$ depends on the value of the parameters a_1, a_2, a_3 and a_4 only through the squared DoP P^2. Note also that the asymptotic variance of $\widehat{P}_4^2$ reaches its maximum for $P^2 = 1/3$.

5.2. Using 2 images

When the intensity vector is $\underline{\boldsymbol{I}} = (I_1, I_2)^T$, the moment estimators of a_1, a_4 and $r = a_2^2 + a_3^2$ can be obtained from the following set of equations:

$$\mathsf{E}\left[I_1\right] = a_1, \ \mathsf{E}\left[I_2\right] = a_4, \quad (13)$$
$$\mathsf{E}\left[I_1 I_2\right] = a_1a_4 + r, \quad (14)$$

The estimators of a_1 and a_4 are directly related to (13):

$$\left(\underline{a}_1, \underline{a}_4\right)^T = \left(\widehat{a}_1, \widehat{a}_4\right)^T,$$

whereas the estimator of r obtained from (**??**) is

$$\underline{r}_{\text{Mo}} = \frac{1}{n}\sum_{j=1}^{n} I_1^j I_2^j - \widehat{a}_1\widehat{a}_4$$

The asymptotic covariance matrix of the moment estimator vector $\underline{\boldsymbol{\eta}}_{2,\text{Mo}} = \left(\widehat{a}_1, \widehat{a}_4, \underline{r}_{\text{Mo}}\right)^T$ can be easily computed as:

$$\text{var}_\text{A}\left(\underline{\boldsymbol{\eta}}_{2,\text{Mo}}\right) = \frac{1}{n}\begin{pmatrix} a_1^2 & r & 2a_1r \\ r & a_4^2 & 2a_4r \\ 2a_1 & 2a_4r & a_1^2a_4^2 + 4a_1a_4r + 3r^2 \end{pmatrix}.$$

The following DoP estimator is finally obtained

$$\underline{P}_{2\ \text{Mo}}^2 = 1 - \frac{4\left[\widehat{a}_1\widehat{a}_4 - \underline{r}_{\text{Mo}})\right]}{(\widehat{a}_1 + \widehat{a}_4)^2}.$$

The asymptotic variance of $\underline{P}_{2\ \text{Mo}}^2$ is expressed as:

$$\begin{aligned} \text{var}_\text{A}\left(\underline{P}_{2\ \text{Mo}}^2\right) &= G_2^T \text{var}_\text{A}\left(\underline{\boldsymbol{\eta}}_{2,\text{Mo}}\right) G_2, \\ &= \frac{2(1-P^2)^2(P^2+1/2)}{n} + \frac{64a_1a_4r}{n(a_1+a_4)^4}, \end{aligned}$$

where G_2 is the gradient of the transformation from (a_1, a_4, r) to P^2, which has been defined previously.

6. SIMULATION RESULTS

Several experiments have been conducted to evaluate the performance of the estimators derived in this paper. The simulations presented here have been obtained with polarimetric images with 9 different DoPs reported in Table 1. The corresponding entries of the covariance matrices of the Jones vector (1), denoted as Γ_i for $i \in \{0, \ldots, 8\}$, are given in Table 2. The sample size is $n = 15 \times 15$ in all simulations. This corresponds to a squared observation window containing 225 pixels.

Table 1. Polarimetric image DoPs.

P_0^2	P_1^2	P_2^2	P_3^2	P_4^2	P_5^2	P_6^2	P_7^2	P_8^2
0	0.2	0.3	0.4	0.5	0.6	0.8	0.9	0.99

Figure 1 shows the log mean square errors (MSEs) of the DoP estimates using two images by the ML method (plus markers) and the method of moments (cross markers).

Table 2. Covariance matrices of the Jones vector.

	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
a_1	2	15	1	16	82	18	30	2	1.25
a_2	0	0.2	0.4	0	0	7	16	0.6	0
a_3	0	0.5	$\sqrt{0.14}$	0	13	8	8	1.8	5.5
a_4	2	6	1	3.6	17	11	14	2	26

These MSEs can be compared to those corresponding to 4 images (diamond markers). The loss of performance obtained when observing two polarimetric images instead of four can be clearly observed (note again that the ML method and the method of moments coincide when 4 images are observed, as explained in section 5.1). The theoretical asymptotic log MSEs of the different estimators are also depicted. The asymptotic MSEs of the different estimators match perfectly with their estimates, except for the MLE associated to the matrices Γ_0, Γ_1 and Γ_3. This can be explained for matrices Γ_0 and Γ_3 by noting that the parameter $r = a_2^2 + a_3^2$ equals zero in these cases. In other words, r belongs to the boundary of its definition domain, preventing the use of its theoretical asymptotic variance [9, chap. 18]. The difference between estimated and theoretical results regarding the matrix Γ_1 can be explained by noting that the parameter r is close to 0. In this case, the asymptotic MSE of the estimator is not reached for this sample size (a better match would be obtained for a larger sample size). A last comment resulting from Fig. 1 is that all estimators reach their best performance for large values of the DoP.

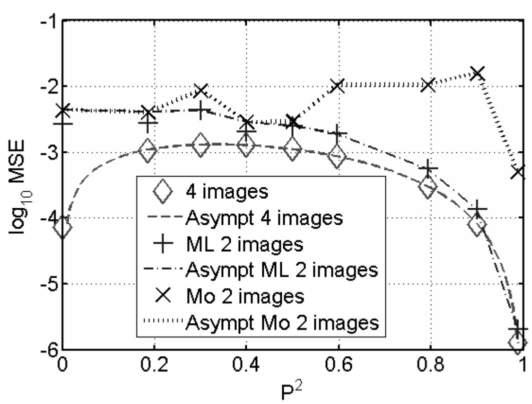

Fig. 1. logMSEs of the DoP estimates using 2 and 4 images vs P^2 ($n = 15 \times 15$).

7. CONCLUSION

This paper showed that the joint distribution of intensity images is a multivarate gamma distribution. This distribution was used to derive ML estimators of the DoP associated to multiple polarimetric images. Simulation results indicated that the DoP can be estimated with two images without significant loss of performance. An extension of this work to low-flux images corrupted by Poisson noise is currently under investigation.

A. A NATURAL EXPONENTIAL FAMILY

Assume that the Jones vector $\boldsymbol{A}$ is complex circular Gaussian with pdf (2). The vector

$$\boldsymbol{A}^r = \left[\Re(A_X), \Re(A_Y), \Im(A_X), \Im(A_Y)\right]^T \in \mathbb{R}^4,$$

is distributed according to a 4 dimensional zero-mean gaussian distribution with covariance matrix:

$$\Sigma = \frac{1}{2}\begin{pmatrix} a_1 & a_2 & 0 & -a_3 \\ a_2 & a_4 & a_3 & 0 \\ 0 & a_3 & a_1 & a_2 \\ -a_3 & 0 & a_2 & a_4 \end{pmatrix}.$$

The pdf of $\boldsymbol{A}^r$ can then be written as:

$$f_{\boldsymbol{A}^r}(\boldsymbol{a}) = \frac{\sqrt{|\eta|}}{\pi^2} \exp\left[-\mathrm{tr}\left(\eta \boldsymbol{a}\boldsymbol{a}^T\right)\right], \qquad (15)$$

where $\eta = \frac{1}{2}\Sigma^{-1}$. Consequently, the rank one matrix $\boldsymbol{U} = \boldsymbol{A}\boldsymbol{A}^T \in \mathbb{R}^{4\times 4}$ has the following pdf:

$$f_{\boldsymbol{U}}(\boldsymbol{u}) = \frac{\sqrt{|\eta|}}{\pi^2} \exp\left[\mathrm{tr}\left(-\eta \boldsymbol{u}\right)\right] \frac{\mathbb{I}_{\Omega_{\boldsymbol{U}}}(\boldsymbol{u})}{u_{1,1}^2}, \qquad (16)$$

where $u_{i,j}$ are the entries of the matrix $\boldsymbol{u}$ for $1 \le i, j \le 4$ and $\Omega_{\boldsymbol{U}} = \{\boldsymbol{u} \in \mathbb{R}^{4\times 4} | u_{1,1} > 0, u_{i,j}u_{1,1} - u_{i,1}u_{j,1} = 0,\ 1 \le i, j \le 4\}$. Looking at (16), we observe that $\boldsymbol{U}$ belongs to a natural exponential family of dimension 4 generated by the measure $\mu(d\boldsymbol{u}) = \frac{\mathbb{I}_{\Omega_{\boldsymbol{U}}}(\boldsymbol{u})d\boldsymbol{u}}{\pi^2 u_{1,1}^2}$ with natural parameter η. Thus the distribution of $\boldsymbol{I}$ also belongs to a natural exponential family, since $\boldsymbol{I}$ and $\boldsymbol{U}$ are related by a one-to-one linear transformation.

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